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Triviality of GHZ operators of higher spin

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Abstract

We prove that local observables of the set of GHZ operators for particles of spin higher than $1/2$ reduce to direct sums of the spin $1/2$ operators σ_x , σ_y and, therefore, no new contradictions with local realism arise by considering them.

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1 Introduction

The GHZ theorem [1] provides a powerful test of quantum non-locality, which can be confirmed or refuted by the outcome of just one single experiment [2]. Formulated for three spin 1/2 particles [2] [3], the argument is based on the anti-commutative nature of the 2x2 spin operators σ_x, σ_y . The values of the three mutually commuting observables

$$\sigma_x^a \otimes \sigma_y^b \otimes \sigma_z^c \equiv \sigma_x^a \sigma_y^b \sigma_z^c, \quad \sigma_y^a \sigma_x^b \sigma_z^c, \quad \sigma_y^a \sigma_y^b \sigma_x^c, \quad (1)$$

and their product, $-\sigma_x^a \sigma_x^b \sigma_x^c$, cannot be obtained, consistently, by making local assignments to each of the individual spin operators, $m_x^I, m_y^I = \pm 1, I = a, b, c$. This is not a contradiction of Quantum Mechanics: the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle)$, for instance, is one of the common eigenstates of the four operators, with eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = 1, \lambda_4 = -1$, respectively. $|\psi\rangle$ is a highly correlated (entangled) state of the three parties which has no defined value for σ_x^I, σ_y^I .

In this note we address the question of how to generalize the argument to particles of higher spin and find that there are no non-trivial extensions other than direct sums of operators that can be brought into the form σ_x, σ_y by means of local unitarity transformations. (For odd dimensional Hilbert spaces the direct sum is completed by a one-dimensional submatrix, i.e., a c-number in the diagonal). We give a proof for the cases of spin 1 and 3/2. Similar problems have been addressed in [4].

Let us look for observables A, B such that $AB = \omega BA$ (their hermiticity implies that ω is at most a phase): this is a necessary condition for the commutator relations $[A_1^a A_2^b A_3^c, B_1^a B_2^b B_3^c] = \text{etc...} = 0$ to hold. As we shall see, all interesting cases correspond to $\omega = -1$. Without loss of generality, A can always be taken diagonal, $A = \text{diag}(\lambda_1, \lambda_2)$, for the simplest case $s=1/2$. The above condition reads

$$AB - \omega BA = \begin{pmatrix} (1 - \omega)\lambda_1 b_{11} & (\lambda_1 - \omega\lambda_2)b_{12} \\ (\lambda_2 - \omega\lambda_1)b_{12}^* & (1 - \omega)\lambda_2 b_{22} \end{pmatrix} = 0. \quad (2)$$

If $\omega \neq 1$, a solution with non-vanishing off-diagonal elements is allowed if $\omega^2 = 1$, i.e., $\omega = -1$. This leads to

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & b \\ b^* & 0 \end{pmatrix}, \quad (3)$$

which can always be transformed to σ_x and σ_y , by rotations and adequate normalization. These are the operators of the example (1). For spin 1/2 the set of GHZ operators are in this sense unique.

2 Spin one

For higher spins the proof proceeds along the same lines. We find one case of interest, with $\omega = -1$,

$$A = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & b & c \\ b^* & 0 & 0 \\ c^* & 0 & 0 \end{pmatrix}. \quad (4)$$

In the basis where B is diagonal A and B read

$$A = - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \sqrt{|b|^2 + |c|^2} \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix} \quad (5)$$

which proves the assertion in the case of spin one, as a rotation around x brings B into the form $0 \oplus \sigma_y$, while A is left as $1 \oplus \sigma_x$, up to normalizations.

3 Spin 3/2

For spin 3/2, in addition to cases that reduce straightforwardly to those of lower spins, we find:

$$A = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & a & b & c \\ a^* & 0 & 0 & 0 \\ b^* & 0 & 0 & 0 \\ c^* & 0 & 0 & 0 \end{pmatrix}. \quad (6)$$

In the basis where B is diagonal A and B read

$$A = - \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B = \sqrt{|a|^2 + |b|^2 + |c|^2} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \quad (7)$$

which is again diagonal in two, 2x2, blocks.

The last case corresponds to

$$A = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & a & 0 & b \\ a^* & 0 & c^* & 0 \\ 0 & c & 0 & d \\ b^* & 0 & d^* & 0 \end{pmatrix}. \quad (8)$$

The following list of unitary transformations bring these matrices to the desired form:

a) With

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (9)$$

$F^\dagger = F = F^{-1}$, we find

$$A' = FAF = \begin{pmatrix} I & \\ & -I \end{pmatrix}, \quad B' = FBF = \begin{pmatrix} & \mathcal{B} \\ \mathcal{B}^\dagger & \end{pmatrix}, \quad (10)$$

where

$$\mathcal{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

b) A unitary transformation of the form $U = \begin{pmatrix} U_1 & \\ & U_2 \end{pmatrix}$ leaves A' invariant and allows to diagonalize \mathcal{B}

$$A'' = A', \quad B'' = UB'U^\dagger = \begin{pmatrix} & U_1 \mathcal{B} U_2^\dagger \\ (U_1 \mathcal{B} U_2^\dagger)^\dagger & \end{pmatrix} = \begin{pmatrix} & m & 0 \\ & 0 & n \\ m^* & 0 & \\ 0 & n^* & \end{pmatrix}. \quad (11)$$

We have used the result that the generic matrix \mathcal{B} can be brought to a diagonal form with two unitary matrices U_1, U_2 .

c) Finally, acting with F again,

$$A''' = A, \quad B''' = \begin{pmatrix} 0 & m & & \\ m^* & 0 & & \\ & & 0 & n \\ & & n^* & 0 \end{pmatrix}, \quad (12)$$

which completes the proof.

4 Conclusions

We conclude that the equation $AB = \omega BA$ is very restrictive on ω and on the possible forms of A and B ; as the Hilbert space dimension increases, with increasing spin, all its solutions for $\omega \neq 1$ have $\omega = -1$ and are essentially direct sums of the two-dimensional σ_x and σ_y . In this sense there are no solutions that could, in principle, enrich the possibilities opened by the GHZ theorem.

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